

Theorem

Statement:

A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Proof:

Suppose that G is an Euler graph. It therefore contains an Euler line (which is a closed walk). In tracing this walk we observe that every time the walk meets a vertex v it goes through two “new” edges incident on v —with one we “entered” v and with the other “exited.” This is true not only of all intermediate vertices of the walk but also of the terminal vertex, because we “exited” and “entered” the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even.

To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; the tracing cannot stop at any vertex but v . And since v is also of even degree, we shall eventually reach v when the tracing comes to an end. If this closed walk h we just traced includes all the edges of G , G is an Euler graph.

If not, we remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges. Since both G and h have all their vertices of even degree, the degrees of the vertices of h' are also even. Moreover, h' must touch h at least at one vertex a , because G is connected. Starting from a , we can again construct a new walk in graph h' . Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a ; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h . This process can be repeated until we obtain a closed walk that traverses all the edges of G . Thus G is an Euler graph. ■